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EXACT SOLUTIONS OF SOME MIXED PROBLEMS OF UNCOUPLED THERMOELASTICITY FOR A TRUNCATED HOLLOW CIRCULAR CONE WITH A GROOVE ALONG THE GENERATRIX†

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An elastic body of finite dimensions in the form of a truncated hollow circular cone with a groove along the generatrix is considered. The uncoupled problem of thermoelasticity is formulated for this body for different types of boundary conditions on all the surfaces. These are the conditions for specifying the displacements or sliding clamping on surfaces with fixed angular coordinates and the conditions for specifying the stresses on surfaces with a fixed radial coordinate (shear stresses are assumed to be zero). It is assumed that the temperature is a specified function of all the spherical coordinates. Some auxiliary functions, related to the displacements, are introduced first, and equations for these functions are then derived using Lamé's equations. A finite integral Fourier transformation with respect to one of the angular variables is then employed. After this, by solving certain Sturm–Liouville problems, a new integral transformation is constructed and is applied to the equations with respect to the other angular variable. As a result a one-dimensional system of differential equations is obtained, to solve which an integral Mellin transformation is employed in a special way. Finally, exact solutions of some problems of thermoelasticity are constructed in series for this body.

1. FORMULATION OF THE PROBLEMS

We consider steady problems of uncoupled thermoelasticity for a body occupying a region described in a spherical system of coordinates (r, θ, φ) by the relations

$$a_0 \leqslant r \leqslant a_1, \quad \omega_0 \leqslant \theta \leqslant \omega_1, \quad \varphi_0 \leqslant \varphi \leqslant \varphi_1 \tag{1.1}$$

It is assumed that the temperature field $T(r, \theta, \varphi)$, obtained from the solution of the fairly simple harmonic boundary-value problem, is known. The displacements u_r, u_θ and u_φ or the conditions of sliding clamping

$$u_\theta|_{\theta=\omega_i} = 0, \quad \tau_{\theta r}|_{\theta=\omega_i} = \tau_{\theta_\varphi}|_{\theta=\omega_i} = 0; \quad i = 0, 1 \tag{1.2}$$

are given on the conical surfaces $\theta = \omega_i$ $(i = 0, 1)$. On the plane surface $\varphi = \varphi_i$ $(i = 0, 1)$ either the displacements are specified or also the conditions for sliding clamping

$$u_\varphi|_{\varphi=\varphi_i} = 0, \quad \tau_{\varphi r}|_{\varphi=\varphi_i} = \tau_{\varphi\theta}|_{\varphi=\varphi_i} = 0; \quad i = 0, 1 \tag{1.3}$$

The conditions may be arbitrary on the spherical surfaces $r = a_i$ $(i = 0, 1)$, but to fix our ideas we will take the conditions of the first fundamental problem

$$\sigma_r|_{r=a_i} = -p_i(\theta, \varphi), \quad \tau_{r\theta}|_{r=a_i} = \tau_{r\varphi}|_{r=a_i} = 0; \quad i = 0, 1 \tag{1.4}$$

We construct exact solutions of these problems below.

2. TRANSFORMATION OF THE THERMOELASTICITY EQUATIONS BY INTRODUCING NEW UNKNOWN FUNCTIONS

Following the approach proposed earlier [1], instead of the displacements (G is the shear modulus)

$$2Gu_r = u, \quad 2Gu_\theta = V, \quad 2Gu_\varphi = W \tag{2.1}$$

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We will introduce the functions

$$\left\| \begin{matrix} Z(r,\theta,\varphi) \\ Z^*(r,\theta,\varphi) \end{matrix} \right\| = \frac{1}{\sin\theta} \left\{ \left\| \begin{matrix} V\sin\theta \\ W\sin\theta \end{matrix} \right\|^{\cdot} \pm \left\| \begin{matrix} W \\ V \end{matrix} \right\|^{,} \right\} \tag{2.2}$$

Here and below derivatives with respect to the variable r will be denoted by a prime, derivatives with respect to θ will be denoted by a dot and derivatives with respect to φ will be denoted by a superscript comma.

The thermoelasticity equations in a spherical system of coordinates, taking relations (2.1) and (2.2) into account, can be written in the form [2.3]

$$\begin{gathered} \Delta U - 2U - 2Z + \frac{r^2}{1-2\mu}\left(\frac{\tilde{Z}}{r}\right)' = \alpha_\mu r^2 T' \\ \Delta V + 2U^{\cdot} - \frac{2\operatorname{ctg}\theta}{\sin^2\theta} W^{,} - \frac{V}{\sin\theta} + \frac{r\tilde{Z}^{\cdot}}{1-2\mu} = \alpha_\mu r T^{\cdot} \\ \Delta W + \frac{2U^{,}}{\sin\theta} - \frac{2\operatorname{ctg}\theta}{\sin\theta} V^{,} - \frac{W}{\sin^2\theta} + \frac{r\tilde{Z}^{,}}{(1-2\mu)\sin\theta} = \frac{\alpha_\mu r T^{,}}{\sin\theta} \end{gathered} \tag{2.3}$$

where

$$\Delta U = (r^2U')' + \nabla U \nabla U = \frac{(\sin\theta U^{\cdot})^{\cdot}}{\sin\theta} + \frac{u^{,,}}{\sin^2\theta}\,\alpha_\mu = 4G\frac{1+\mu}{1-2\mu}\alpha_T \tag{2.4}$$

$$\tilde{Z} = \frac{(r^2U)'}{r^2} + Z$$

(μ is Poisson's ratio and α_T is the coefficient of linear expansion).

We will subject the second and third equations of system (2.3) to a further transformation, for which we multiply the second equation by $\sin\theta$, differentiate with respect to θ and divide by $\sin\theta$, then we differentiate the third equation with respect to φ and divide by $\sin\theta$ and add the equations obtained. We then carry out the same operation on the third equation of (2.3) as was carried out on the second in the previous case, and we carry out the same operation on the second equation as was carried out on the third in the previous case. As a result, instead of system (2.3) we will have

$$\begin{gathered} \Delta U - 2(U+Z) + \mu_0[(r^2U')' + rZ' - Z\} = \alpha_\mu r^2 T' \\ \Delta Z + 2\nabla U + \mu_0[r^{-1}(\nabla U r^2)' + \nabla Z] = \alpha_\mu r \nabla T, \quad \Delta Z^* = 0 \end{gathered} \tag{2.5}$$

If the functions $Z(r, \theta, \varphi)$ and $Z^*(r, \theta, \varphi)$ are obtained, it can be shown [1] that the functions $V(r, \theta, \varphi)$ and $W(r, \theta, \varphi)$ can be obtained from the equations

$$\nabla\left(\sin\theta \left\| \begin{matrix} V \\ W \end{matrix} \right\|\right) = \frac{1}{\sin\theta}\left(\sin^2\theta \left\| \begin{matrix} Z \\ Z^* \end{matrix} \right\|\right)^{\cdot} \mp \left\| \begin{matrix} Z^* \\ Z \end{matrix} \right\|^{,} \tag{2.6}$$

3. INTEGRAL TRANSFORMATION OF THE EQUATIONS OBTAINED WITH RESPECT TO THE VARIABLE φ

The realization of the integral transformation with respect to the variable φ depends on what boundary conditions are imposed on the faces $\varphi = \varphi_i$ ($i = 0, 1$). If the displacements are specified on these, the following boundary conditions are specified for Eqs (2.5)

$$U(r,\theta,\varphi_i) = Z(r,\theta,\varphi_i) = Z^*(r,\theta,\varphi_i) = 0, \quad i = 0,1 \tag{3.1}$$

$$U(r,\omega_i,\varphi) = Z(r,\omega_i,\varphi) = Z^*(r,\omega_i,\varphi) = 0, \quad i = 0,1 \tag{3.2}$$

If the sliding clamping conditions (1.3) and (1.2) are specified, then, instead of conditions (3.1) and (3.2), we take the following

$$U^{\cdot}(r,\theta,\varphi_i)=Z^{\cdot}(r,\theta,\varphi_i)=Z^{*\cdot}(r,\theta,\varphi_i)=0,\quad i=0,1 \tag{3.3}$$

$$U^{\cdot}(r,\omega_i,\varphi)=Z^{\cdot}(r,\omega_i,\varphi)=Z^{*\cdot}(r,\omega_i,\varphi)=0,\quad i=0,1 \tag{3.4}$$

In order for conditions (3.1) to be satisfied, we must apply the following integral transformation [4] to Eqs (2.5) (assuming $\varphi_0 = 0$)

$$X_n(r,\theta)=\int_0^{\varphi_1} X(r,\theta,\varphi)\sin\mu_n\varphi d\varphi,\qquad \mu_n=\frac{n\pi}{\varphi_1},\qquad n=1,2,\ldots \tag{3.5}$$

with the inversion formula [4]

$$X(r,\theta,\varphi)=\frac{2}{\varphi_1}\sum_{n=1}^{\infty}X_n(r,\theta)\sin\mu_n\varphi \tag{3.6}$$

Here and henceforth we will use the following notation

$$X_n(r,\theta)=\left\|\begin{matrix}U_n(r,\theta)\\ Z_n(r,\theta)\end{matrix}\right\|,\qquad X(r,\theta,\varphi)=\left\|\begin{matrix}U(r,\theta,\varphi)\\ Z(r,\theta,\varphi)\end{matrix}\right\|\ \text{etc.}$$

We take similar formulae and the same inversion formulae for $Z_n^*(r, \theta)$ and $T_n(r, \theta)$. Instead of Eqs (2.5) we will then obtain

$$\begin{gathered}(r^2U_n')'-\mu_*^{-1}[\nabla_n^*U_n+2U_n+2\mu'Z_n-\mu_0rZ_n'-\alpha_\mu r^2T_n']=0\\ (r^2Z_n')'-\mu_*\nabla_n^*Z_n-2\mu_*\nabla_n^*U_n-\mu_0\nabla_n^*(rU_n')=-\alpha_\mu r\nabla_n^*T_n\\ (r^2Z_n^{*\prime})'-\nabla_n^*Z_n^*=0;\quad \mu_*=2(1-\mu)\mu_0,\quad \mu'=(3-4\mu)\mu_0\end{gathered} \tag{3.7}$$

where

$$\nabla_n^*f(r,\theta)=\mu_n^2\operatorname{cosec}^2\theta f(r,\theta)-\operatorname{cosec}\theta[f^{\cdot}(r,\theta)\sin\theta]^{\cdot} \tag{3.8}$$

In order for condition (3.3) to be satisfied,it is necessary to use the following integral transformation [4] instead of integral transformation (3.5)

$$X_n(r,\theta)=\int_0^{\varphi_1} X(r,\theta,\varphi)\cos\mu_n\varphi d\varphi,\qquad \mu_n=\frac{(n-1)\pi}{\varphi_1},\qquad n=1,2\ldots \tag{3.9}$$

with the inversion formula [4]

$$X(r,\theta,\varphi)=\frac{1}{\varphi_1}X_1(r,\theta)+\frac{2}{\varphi_1}\sum_{n=2}^{\infty}X_n(r,\theta)\cos\mu_n\varphi \tag{3.10}$$

Similar formulae hold for $Z_n^*(r, \theta)$ and $T_n(r, \theta)$.

Applying integral transformation (3.9) to Eqs (2.5), we again arrive at the same Eqs (3.7), in which for μ_n we must use the formula from (3.9) instead of the formula from (3.5). Boundary conditions (3.2) and (3.4) in transformants (3.5) and (3.9) respectively reduce to the following

$$U_n(r,\omega_i)=Z_n(r,\omega_i)=Z_n^*(r,\omega_i)=0,\quad i=0,1 \tag{3.11}$$

$$U_n^{\cdot}(r,\omega_i)=Z_n^{\cdot}(r,\omega_i)=Z_n^{*\cdot}(r,\omega_i)=0,\quad i=0,1 \tag{3.12}$$

If $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ in relations (1.1) and therefore the cone is continuous in the direction of the variable φ, then instead of integral transformation (3.5) and (3.9) we must use the integral transformation

$$X_n(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(r,\theta,\varphi)e^{-in\varphi}d\varphi, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{3.13}$$

with the inversion formula

$$X(r,\theta,\varphi) = \sum_{n=-\infty}^{\infty} X_n(r,\theta)e^{in\varphi} \tag{3.14}$$

In transforms (3.13), Eqs (2.5) can also be written in the form (3.7), except that in formula (3.8) we must put $\mu_n = |n|$.

4. INTEGRAL TRANSFORMATION WITH RESPECT TO THE VARIABLE θ AND REDUCTION OF THE EQUATIONS OBTAINED TO ONE-DIMENSIONAL EQUATIONS

In order to carry out an integral transformation with respect to the variable θ and simultaneously satisfy boundary conditions (3.11) and (3.12), it is necessary to use an integral transformation whose kernel is the eigenfunction of one of the following Sturm–Liouville problems

$$-\nabla_n^* T_j(\theta) + \tfrac{1}{4} T_j(\theta) = \lambda^2 T_j(\theta), \quad \omega_0 < \theta < \omega_1; \quad l_i^j T_j(\theta) = 0; \quad i, j = 0, 1 \tag{4.1}$$

where

$$l_i^0 T_0(\theta) = T_0(\omega_i), \quad l_i^1 T_1(\theta) = T_1'(\omega_i) + h_i T_1(\omega_i); \quad i = 0, 1 \tag{4.2}$$

However, these boundary-value problems were solved in [5] for the case when $\mu_n = m$ and m are positive integers. The extension of the results obtained earlier in [5] to the case of positive non-integer numbers μ_n can be carried out fairly simply using the same scheme, and we will therefore only present the final results. As previously [5], we will change from the eigenvalues $\lambda_k^{(j)}$ ($j = 0, 1$; $k = 0, 1, 2, \ldots$) of boundary-value problems (4.1) to the eigenvalues $\nu_k^{(j)} = -\tfrac{1}{2} + \lambda_k^{(j)}$ ($j = 0, 1$; $k = 0, 1, 2, \ldots$). In this case the differential equation from (4.1) becomes a Legendre equation and the eigenfunctions of the boundary-value problems will have the form ($j = 0, 1$)

$$y_i(\theta,\nu) = P_\nu^\mu(\cos\theta) l_1^j Q_\nu^\mu - Q_\nu^\mu(\cos\theta) l_1^j P_\nu^\mu, \quad \mu = \mu_n, \quad \nu = \nu_k^{(j)}, \quad k = 0, 1, 2, \ldots \tag{4.3}$$

where $P_\nu^\mu(\cos\theta)$, $Q_\nu^\mu(\cos\theta)$ are Legendre functions of the first and second kind on the cut [6], and the eigenvalues $\nu_k^{(j)}$ ($j = 0, 1$; $k = 0, 1, 2, \ldots$) must be found from the equations

$$\Omega_{\nu,j}^\mu \equiv l_0^j P_\nu^\mu l_1^j Q_\nu^\mu - l_0^j Q_\nu^\mu l_1^j P_\nu^\mu = 0, \quad \mu = \mu_n, \quad \nu = \nu_k^{(j)} \tag{4.4}$$

In particular, when $j = 0$ we will have

$$\begin{gathered}\Omega_{\nu,0}^\mu = P_\nu^\mu(\cos\omega_0) Q_\nu^\mu(\cos\omega_1) - Q_\nu^\mu(\cos\omega_0) P_\nu^\mu(\cos\omega_1) = 0 \\ \nu = \nu_k^0, \quad k = 0, 1, \ldots, \quad \mu = \mu_n = \pi n \varphi_1^{-1}, \quad n = 1, 2, \ldots\end{gathered} \tag{4.5}$$

and the integral transformations obtained [5] for the case when $\mu_n = m$ (and written in the form of formulae (2.41), [5]) now take the form

$$\begin{gathered} g_k^{(j)} = \int_{\omega_0}^{\omega_1} g(\theta) y_j(\theta,\nu)\sin\theta d\theta, \quad j = 0, 1, \quad \nu = \nu_k^j \\ g(\theta) = \sum_{k=0}^{\infty} \frac{g_k^{(j)} y_j(\theta,\nu)}{\|y_j(\theta,\nu)\|^2} = -\sum_{k=0}^{\infty} \frac{g_k^{(j)} y_j(\theta,\nu)}{\sigma_{\mu k}^{(j)}(\omega_0,\omega_i)}, \quad \mu = \mu_n \end{gathered} \tag{4.6}$$

where

$$\|y_j(\theta,\nu)\|^2 = \int_{\omega_0}^{\omega_1} y_j^2(\theta,\nu)\sin\theta d\theta, \quad j = 0, 1, \quad \nu = \nu_k^j \tag{4.7}$$

$$-\sigma^0_{\mu k} = \frac{Q^\mu_\nu(\cos\omega_1)\Gamma_{\mu,\nu}}{Q^\mu_\nu(\cos\omega_0)(2\nu+1)}\frac{\partial}{\partial\nu}\Omega\frac{\mu}{\nu,0} = \frac{P^\mu_\nu(\cos\omega_1)\Gamma_{\mu,\nu}}{Q^\mu_\nu(\cos\omega_0)(2\nu+1)}\frac{\partial}{\partial\nu}\Omega\frac{\mu}{\nu,0}, \quad \mu=\mu_n, \quad \nu=\nu^0_k \tag{4.8}$$

$$-\sigma^1_{\mu k} = \frac{l^1_1 Q^\mu_\nu\Gamma_{\mu\nu}}{l^1_0 Q^\mu_\nu(2\nu+1)}\frac{\partial}{\partial\nu}\Omega^\mu_{\nu,1} = \frac{l^1_1 P^\mu_\nu\Gamma_{\mu\nu}}{l^1_0 P^\mu_\nu(2\nu+1)}\frac{\partial}{\partial\nu}\Omega^\mu_{\nu,1}, \quad \mu=\mu_n, \quad \nu=\nu^1_k \tag{4.9}$$

The second equations in (4.8) and (4.9) hold by virtue of relations (4.5) and (4.4). For $\Gamma_{\mu\nu}$ formula (2.6) from [5] holds when m is replaced by $\mu = \mu_n$.

To reduce Eqs (3.7) to one-dimensional equations while satisfying boundary conditions (3.11), we will use integral transformation (4.6) with $j = 0$, i.e. we change in (3.7) to the transforms

$$X_{nk}(r) = \int_{\omega_0}^{\omega_1} X_n(r,\theta)y_0(\theta,\nu)\sin\theta d\theta, \quad \nu=\nu^j_k \tag{4.10}$$

where the eigenfunctions $y_0(\theta, \nu)$, according to relations (4.3) and (4.6), will have the form

$$y_0(\theta,\nu) = P^\mu_\nu(\cos\theta)Q^\mu_\nu(\cos\omega_1) - Q^\mu_\nu(\cos\theta)P^\mu_\nu(\cos\omega_1) = 0 \tag{4.11}$$
$$\nu=\nu^0_k, \quad k=0,1,\ldots, \quad \mu=\mu_n=\pi n\varphi_1^{-1}, \quad n=1,2,\ldots$$

and satisfy the boundary condition

$$y_0(\omega_i,\nu)=0, \quad \nu=\nu^0_k, \quad i=0,1 \tag{4.12}$$

The inversion formula for transform (4.6), by the second relation of (4.6), can be written in the form

$$X_n(r,\theta) = -\sum_{k=0}^{\infty} X_{nk}(r)\frac{y_0(\theta,\nu^0_k)}{\sigma^0_{\mu k}(\omega_0,\omega_1)} \tag{4.13}$$

Formulae similar to (4.10) and (4.13) hold for the transforms $Z^*_{nk}(r)$ and $T_{nk}(r)$.

In transforms (4.10) we write Eqs (3.7) in the form

$$[r^2U'_{nk}(r)]' - (2+N_\nu\mu_*^{-1})U_{nk} - \mu'\mu_*^{-1}Z_{nk} + \mu_0\mu_*^{-1}rZ'_{nk} = \alpha_\mu\mu_*^{-1}r^2T'_{nk}(r)$$
$$[r^2Z'_{nk}(r)]' - N_\nu\mu_*Z_{nk} - \mu_0N_\nu rU'_{nk} - 2\mu_*N_\nu U_{nk} = -\alpha_\mu rN_\nu T_{nk}(r) \tag{4.14}$$
$$[r^2Z^{*\prime}_{nk}(r)]' - N_\nu Z^*_{nk} = 0, \quad a_0<r<a_1; \quad N_\nu=\nu(\nu+1); \quad \nu=\nu^0_k, \quad k=0,1,\ldots$$

In order to reduce Eqs (3.7) to one-dimensional forms and to satisfy conditions (3.12), we must use integral transformation (4.6) with $j = 1$, putting $h_i = 0$ ($i = 0, 1$) there. Then the eigenfunction $y_*(\theta, \nu) = y_1(\theta, \nu)|_{h_i=0}$, by relations (5.3) and (4.2), takes the form

$$y_*(\theta,\nu) = P^\mu_\nu(\cos\theta)\frac{d}{d\omega_1}Q^\mu_\nu(\cos\omega_1) - Q^\mu_\nu(\cos\theta)\frac{d}{d\omega_1}P^\mu_\nu(\cos\omega_1) \tag{4.15}$$

Here $\mu_1 = \mu_n$ and the expression for μ_n is taken from (3.9), and we must take as ν the eigenvalues ν^*_k ($k = 0, 1, 2, \ldots$), which must be found, according to relations (4.4) and (4.2), from the transcendental equation

$$\Omega^\mu_\nu \equiv \frac{dP^\mu_\nu(\cos\omega_0)}{d\omega_0}\frac{dQ^\mu_\nu(\cos\omega_1)}{d\omega_1} - \frac{dQ^\mu_\nu(\cos\omega_0)}{d\omega_0}\frac{dP^\mu_\nu(\cos\omega_1)}{d\omega_1} = 0 \tag{4.16}$$
$$\nu=\nu^*_k, \quad k=0,1,\ldots; \quad \mu=\mu_n=\pi(n-1)\varphi_1^{-1}, \quad n=1,2,\ldots$$

If we now introduce the transforms

$$X_{nk}(r) = \int_{\omega_0}^{\omega_1} X_n(r,\theta)y_*(\theta,\nu)\sin\theta d\theta, \quad \nu=\nu^*_k, \quad k=0,1,\ldots \tag{4.17}$$

then, in these transforms (for $Z^*_{nk}(r)$ and $T_{nk}(r)$ the formulae are analogous) Eqs (3.7) are converted into one-dimensional equations (4.14), in which the parameters ν and μ_n must be taken as in (4.16).

The inversion formulae for transforms (4.17), by relations (4.7), (4.9) and (4.16), can be written in the form

$$X_n(r,\theta) = -\sum_{k=0}^{\infty} X_{nk}(r)\frac{y_*(\theta,\nu_k^*)}{\sigma^*_{\mu k}(\omega_0,\omega_1)} \tag{4.18}$$

$$-\sigma^*_{\mu k} = \frac{\Gamma_{\mu,\nu}dP_\nu^\mu(\cos\omega_1)/d\omega_1}{(2\nu+1)dP_\nu^\mu(\cos\omega_0)/d\omega_0}\frac{d}{d\nu}\Omega_\nu^\mu$$

$$\nu=\nu_k^*,\quad k=0,1,\ldots,\qquad \mu=\mu_n=\pi(n-1)\varphi_1^{-1},\qquad n=1,2,\ldots$$

When $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ in relations (1.1), we arrive at the same one-dimensional equations (4.14) in which $\mu_n = |n|$ and $\nu = \nu_k^a$ or $\nu = \nu_k^c$, depending on whether the faces $\theta = \omega_i$ ($i = 0, 1$) are rigidly clamped or there is a sliding clamping. Here, in the first case, transforms (4.10) are taken in the accordance with the formulae

$$X_{nk}(r) = \int_{\omega_0}^{\omega_1} X_n(r,\theta)\varphi_a^m(\theta,\nu)\sin\theta d\theta,\qquad \nu=\nu_k^a,\quad k=0,1,\ldots,\quad m=|n| \tag{4.19}$$

The originals of these transforms are found [5] from the formulae

$$X_n(r,\theta) = -\sum_{k=0}^{\infty} X_{nk}(r)\frac{\varphi_a^m(\theta,\nu)}{\sigma^a_{mk}(\omega_0,\omega_1)},\qquad \nu=\nu_k^a,\quad k=0,1,2,\ldots \tag{4.20}$$

In formulae (4.19) and (4.20) $\varphi_a^m(\theta, \nu)$ is taken in accordance with formula (3.24), while $\sigma^a_{mk}(\omega_0, \omega_1)$ is taken in accordance with formula (2.14) from [5]; the eigenvalues ν_k^a ($k = 0, 1, 2, \ldots$) are found from transcendental equation (2.27) [5]. The formulae are analogous for the transforms $Z^*_{nk}(r)$ and $T_{nk}(r)$. In the second case, transforms (4.17) are taken in accordance with the formulae

$$X_{nk}(r) = \int_{\omega_0}^{\omega_1} X_n(r,\theta)\varphi_c^m(\theta,\nu)\sin\theta d\theta,\qquad \nu=\nu_k^c,\quad k=0,1,\ldots,\quad m=|n| \tag{4.21}$$

for which the inversion formulae have the form [5]

$$X_n(r,\theta) = -\sum_{k=0}^{\infty} X_{nk}(r)\frac{\varphi_c^m(\theta,\nu)}{\sigma^c_{mk}(\omega_0,\omega_1)},\qquad \nu=\nu_k^c,\quad k=0,1,2,\ldots \tag{4.22}$$

where $\varphi_c^m(\theta, \nu)$ and $\sigma^c_{mk}(\omega_0, \omega_1)$ are taken in accordance with formulae (2.37) and (2.49) from [5], while the eigenvalues ν_k^c ($k = 0, 1, 2, \ldots$) are the roots of Eq. (2.39) from [5].

5. FORMULATION OF THE ONE-DIMENSIONAL BOUNDARY-VALUE PROBLEMS FOR THE AUXILIARY FUNCTIONS AND THEIR SOLUTION

To formulate the one-dimensional boundary-value problems for the auxiliary functions we must add to the system of equations (4.14) the boundary conditions at the points $r = a_i$ ($i = 0, 1$). We obtain these boundary conditions by realizing conditions (1.4). In order to do this, as previously [1], we introduce combinations of shear stresses

$$\left\|\begin{matrix}\tau(r,\theta,\varphi)\\ \tau^*(r,\theta,\varphi)\end{matrix}\right\| = \frac{1}{\sin\theta}\left\{\left\|\begin{matrix}\sin\theta\tau_{r\theta}\\ \sin\theta\tau_{r\varphi}\end{matrix}\right\| \pm \left\|\begin{matrix}\tau_{r\varphi}\\ \tau_{r\theta}\end{matrix}\right\|\right\} \tag{5.1}$$

Using the formulae

$$2\tau_{r\theta} = r\left(\frac{V}{r}\right)' + \frac{U^{\cdot}}{r},\qquad 2\tau_{r\varphi} = \frac{U^{\cdot}}{r\sin\theta} + r\left(\frac{W}{r}\right)' \tag{5.2}$$

$$\sigma_r = \frac{2\mu U + (1-\mu)rU' + \mu Z}{(1-2\mu)r} - \alpha_\mu T$$

which follow from Hooke's law in a spherical system of coordinates [2, 3], we obtain

$$2r\tau = \nabla U + rZ' - Z, \quad 2r\tau_* = rZ^{*'} - Z^* \tag{5.3}$$

According to (4.1), the second and third conditions from (1.4) will be satisfied if

$$\tau(r, \omega_i, \varphi) = \tau^*(r, \omega_i, \varphi) = 0, \quad i = 0,1 \tag{5.4}$$

Using the third formula of (5.2) we can write the first boundary condition from (1.4) in the form

$$\begin{gathered} 2\mu U(r, \omega_i, \varphi) + (1-\mu) rU'(r, \omega_i, \varphi) + \mu Z(r, \omega_i, \varphi) = -(1-2\mu)a_i q_i(\theta, \varphi), \\ q_i(\theta, \varphi) = p_i(\theta, \varphi) - \alpha_\mu T(a_i, \theta, \varphi), \quad i = 0, 1 \end{gathered} \tag{5.5}$$

If we apply integral transformations (3.5), (3.9) and (3.12) to relations (5.3)–(5.5), and then (4.10), (4.17) and (4.19), boundary conditions (1.4) can be written in the form

$$\begin{gathered} N_\nu U_{nk}(a_i) + Z_{nk}(a_i) - a_i Z'_{nk}(a_i) = 0, \quad a_i Z^{*'}_{nk}(a_i) - Z^*_{nk}(a_i) = 0, \quad i = 0, 1 \\ 2\mu U_{nk}(a_i) - (1-\mu) a_i U'_{nk}(a_i) + \mu Z_{nk}(a_i) = -(1-2\mu) a_i q_{ink}, \quad i = 0, 1 \end{gathered} \tag{5.6}$$

Hence it follows that $Z_{nk}(r)$ satisfies the homogeneous boundary-value problem, and consequently

$$Z^*_{nk}(r) \equiv 0, \quad Z^*(r, \theta, \varphi) \equiv 0 \tag{5.7}$$

To set up the boundary-value problem for the functions $U_{nk}(r)$ and $Z_{nk}(r)$ it is convenient to introduce the system of functions

$$\begin{gathered} y_0(r) = u_{nk}(r), \quad y_1(r) = rU'_{nk}(r), \quad y_2(r) = Z_{nk}(r), \quad y_3(r) = rZ'_{nk}(r) \\ f_1(r) = \mu_*^{-1} r^2 T_{nk}(r), \quad f_2(r) = -N_\nu r T_{nk}(r) \end{gathered} \tag{5.8}$$

If we then take into account the fact that

$$ry'_0(r) = y_1(r), \quad ry'_2(r) = y_3(r) \quad (r^2 U'_{nk})' = r(rU'_{nk})' + rU'_{nk}$$

and introduce the vectors and matrices

$$\mathbf{y}(r) = \begin{Vmatrix} y_0(r) \\ y_1(r) \\ y_2(r) \\ y_3(r) \end{Vmatrix}, \quad \mathbf{f}(r) = \begin{Vmatrix} 0 \\ f_1(r) \\ 0 \\ f_2(r) \end{Vmatrix}, \quad \boldsymbol{\gamma} = \begin{Vmatrix} 0 \\ 0 \\ a_0 q_{0nk} \\ a_1 q_{1nk} \end{Vmatrix} \tag{5.9}$$

$$\mathbf{P}_k = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 2 + \mu_*^{-1} N_\nu & -1 & \mu_*^{-1}\mu' & -\mu_*^{-1}\mu_0 \\ 0 & 0 & 0 & 1 \\ 2\mu_* N_\nu & \mu_0 N_\nu & \mu_* N_\nu & -1 \end{Vmatrix}, \quad \mathbf{A} = \begin{Vmatrix} N_\nu & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 2\mu & 1-\mu & \mu & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{5.10}$$

and also the matrix B, which is obtained from A by interchanging the first two and the last two rows, then, to find the functions $U_{nk}(r)$ and $Z_{nk}(r)$ we arrive at the vector boundary-value problem

$$\begin{gathered} r\mathbf{y}'(r) - \mathbf{P}_k \mathbf{y}(r) = \alpha_\mu \mathbf{f}(r), \quad a_0 < r < a_1 \\ \mathbf{U}[\mathbf{y}(r)] \equiv \mathbf{A}\mathbf{y}(a_0) + \mathbf{B}\mathbf{y}(a_1) = -(1-2\mu)\boldsymbol{\gamma} \end{gathered} \tag{5.11}$$

To solve the differential equation from (5.11) we apply to it the Mellin integral transformation

$$\begin{Vmatrix} \mathbf{y}_s \\ \mathbf{f}_s \end{Vmatrix} = \int_0^\infty r^{s-1} \begin{Vmatrix} \mathbf{y}(r) \\ \mathbf{f}(r) \end{Vmatrix} dr$$

first extending the right-hand by zero up to the interval $(0, \infty)$. As a result we obtain

$$\mathbf{y}(r) = \alpha_\mu \int_{a_0}^{a_1} \mathbf{\Phi}\left(\frac{r}{\rho}\right)\mathbf{f}(\rho)\frac{d\rho}{\rho}, \qquad a_0 \leq r \leq a_1$$

where $\rho^{-1}\mathbf{\Phi}(r/\rho)$ is the fundamental matrix-function [7, 8] of differential equation (5.11) defined by the formula

$$\mathbf{\Phi}(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} (-s\mathbf{I} - \mathbf{P}_k)^{-1} x^{-s} ds = \frac{1}{2\pi i}\int_{-\gamma - i\infty}^{-\gamma + i\infty} (\xi\mathbf{I} - \mathbf{P}_k)^{-1} x^{\xi} d\xi, \quad x = \frac{r}{\rho} \tag{5.12}$$

To calculate the last integral we bear in mind that [9, 7]

$$\begin{gathered}(\xi\mathbf{I} - \mathbf{P}_k)^{-1} = \Delta_k^* Q_4^{-1}(\xi), \quad \mathbf{I}Q_4(\xi) = (\xi\mathbf{I} - \mathbf{P}_k)\Delta_k^* \\ Q_4(\xi) = \det(\xi\mathbf{I} - \mathbf{P}_k) = \prod_{j=1}^{4}(\xi - \xi_j) = \xi^4 + 2\xi^3 - \\ -(2N_\nu + 1)\xi^2 - 2(N_\nu + 1)\xi + N_\nu(N_\nu - 2), \quad \nu = \nu_k, \quad k = 0, 1, 2, \ldots\end{gathered} \tag{5.13}$$

where ν_k are the roots of transcendental equations (4.5) and (4.16) or Eqs (2.27) and (2.39) from [5]. Here the roots of the characteristic polynomial $Q_4(\xi)$ will be defined by the formulae

$$\xi_1 = -2 - \nu_k, \quad \xi_2 = -1 + \nu_k, \quad \xi_3 = -\nu_k, \quad \xi_4 = 1 + \nu_k \tag{5.14}$$

We will represent the characteristic matrix Δ_k^* in the form [9, 7]

$$\Delta_k^*(\xi) = \sum_{j=0}^{3}\xi^j \Delta_{3-j}^{(k)} \tag{5.15}$$

The numerical matrices $\Delta_i^{(k)}$ $(i = 0, 1, 2, 3)$ are obtained by substituting expression (5.15) into the second equality of (5.13) and equating coefficients of powers of ξ. As a result we obtain

$$\begin{gathered}\Delta_0^{(k)} = \mathbf{I}, \quad \Delta_1^{(k)} = 2\mathbf{I} + \mathbf{P}_k, \quad \Delta_2^{(k)} = 2\mathbf{P}_k + \mathbf{P}_k^2 - (2N_\nu + 1)\mathbf{I} \\ \Delta_3^{(k)} = 2\mathbf{P}_k^2 + \mathbf{P}_k^3 - (2N_\nu + 1)\mathbf{P}_k - 2(N_\nu + 1)\mathbf{I}\end{gathered}$$

where we also have the following formula for $\Delta_3^{(k)}$

$$\Delta_3^{(k)} = -N_\nu(N_\nu - 2)\mathbf{P}_k^{-1}, \quad \nu = \nu_k, \quad k = 0, 1, 2, \ldots$$

which can serve to monitor the calculations.

From relations (5.13) and (5.15) we have

$$(\xi\mathbf{I} - \mathbf{P}_k)^{-1} = \sum_{j=0}^{3}\Delta_{3-j}^{(k)}\frac{\xi^j}{Q_4(\xi)}$$

Substituting this expression into (5.12), we obtain

$$\begin{gathered}\mathbf{\Phi}(x) = \sum_{j=0}^{3}\Delta_{3-j}^{(k)}\varphi_j(x), \quad \varphi_j(x) = \left(r\frac{d}{dr}\right)^j \varphi(x) \\ \varphi(x) = \frac{1}{2\pi i}\int_{-\gamma - i\infty}^{-\gamma + i\infty}\frac{x^\xi}{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)}d\xi, \quad x = \frac{r}{\rho}\end{gathered} \tag{5.16}$$

If $-\gamma > -\nu_k$ and $\nu_k > 1$ $(k = 0, 1, 2, \ldots)$, then, from the theorem on residues, taking relation (5.14) and the second equation of (5.16) into account, we obtain

$$\varphi_j(x) = \frac{1}{2(2\nu + 1)}\left[\frac{1}{2\nu - 1}\left\{\begin{matrix}(-\nu)^j x^{-\nu}, & x > 1 \\ (\nu - 1)^j x^{\nu - 1}, & x < 1\end{matrix}\right\} - \frac{1}{2\nu + 3}\left\{\begin{matrix}(-2 - \nu)^j x^{-\nu}, & x > 1 \\ (\nu - 1)^j x^{\nu - 1}, & x < 1\end{matrix}\right\}\right] \tag{5.17}$$

$$j = 0, 1, 2, 3; \quad \nu = \nu_k, \quad k = 0, 1, 2, \ldots$$

In order to obtain a solution of boundary-value problem (5.11) with non-zero boundary conditions, it is necessary [7, 8] to construct Green's matrix $\mathbf{G}(r, \rho)$ and the basis matrix function $\mathbf{\Psi}(r)$. We will start by constructing the latter. To do this [7, 8] we must first solve the matrix differential equation

$$r\mathbf{Z}'(r) - \mathbf{P}_k\mathbf{Z}(r) = 0 \tag{5.18}$$

Using Cauchy's theorem, it can be shown that the matrix (Γ is a closed contour, enveloping all the zeros of the function $Q_4(\xi)$)

$$\mathbf{Z}(r) = \frac{1}{2\pi i}\oint_\Gamma (\mathbf{I}\xi - \mathbf{P}_k)^{-1} r^\xi d\xi = \frac{1}{2\pi i}\oint_\Gamma \frac{\Delta_k^*(\xi)}{Q_4(\xi)} r^\xi d\xi \tag{5.19}$$

is [7, 8] a solution of Eq. (5.18). Substituting expression (5.15) into (5.19) and taking expressions (5.14) into account, as when evaluating the integral (5.12), we arrive at the formula

$$\mathbf{Z}(r) = \sum_{j=0}^{3} \Delta_{3-j}^{(k)} W_j(r), \quad 2(2\nu+1)W_j(r) = \frac{(-\nu)^j r^{-\nu} - (\nu-1)^j r^{\nu-1}}{2\nu-1} +$$

$$+\frac{(\nu+1)^j r^{\nu+1} - (-1)^j(\nu+2)^j r^{-\nu-2}}{2\nu+3}, \quad j = 0,1,2,3; \quad \nu = \nu_k, \quad k = 0,1,\ldots \tag{5.20}$$

Since the matrix $\mathbf{\Psi}(r)$ must satisfy the boundary-value problem [7, 8]

$$r\mathbf{\Psi}'(r) - P_k\mathbf{\Psi}(r) = 0, \quad a_0 < r < a_1, \quad \mathbf{U}[\mathbf{\Psi}(r)] = \mathbf{I} \tag{5.21}$$

it can be shown by a direct check that

$$\mathbf{\Psi}(r) = \mathbf{Z}(r)(\mathbf{U}[\mathbf{Z}(r)])^{-1} \tag{5.22}$$

It can also be shown by a direct check [7, 8], that the matrix

$$\mathbf{G}(r,\rho) = \rho^{-1}\left\{\mathbf{\Phi}\left(\frac{r}{\rho}\right) - \mathbf{\Psi}(r)\mathbf{U}\left[\mathbf{\Phi}\left(\frac{r}{\rho}\right)\right]\right\} \tag{5.23}$$

satisfies all the conditions imposed on Green's matrix of boundary-value problem (5.11). Hence, the solution of the latter can be written in the form

$$\mathbf{y}(r) = \alpha_\mu \int_{a_0}^{a_1} \mathbf{G}(r,\rho)\mathbf{f}(\rho)d\rho - (1-2\mu)\mathbf{\Psi}(r)\gamma, \quad a_0 < r < a_1 \tag{5.24}$$

6. FINDING THE DISPLACEMENT FIELD FROM THE AUXILIARY FUNCTIONS

Using expressions (5.8), formula (5.24) enables us to find the transforms of the displacement $u_r = U(r, \theta, \varphi)(2G)^{-1}$ and of the auxiliary function $Z(r, \theta, \varphi)$. In order to obtain the transforms of the remaining displacements $u_\theta = V(r, \theta, \varphi)(2G^{-1})$, $u_\varphi = W(r, \theta, \varphi)(2G^{-1})$ we will proceed from differential equations (2.7) taking identities (5.7) into account. We will then formulate the boundary conditions for these equations such that the conditions for there to be no displacements on the faces $\varphi = \varphi_i$, $\theta = \omega_i$ (i = 0, 1) or the sliding-clamping conditions (1.2) and (1.3) are completely satisfied.

When solving Eqs (2.17) it is convenient to introduce the notation

$$Y^*(r,\theta,\varphi) = \sin\theta Y(r,\theta,\varphi); \quad Y^*(r,\theta,\varphi) = \begin{Vmatrix} V^*(r,\theta,\varphi) \\ W^*(r,\theta,\varphi) \end{Vmatrix}, \quad Y(r,\theta,\varphi) = \begin{Vmatrix} V(r,\theta,\varphi) \\ W(r,\theta,\varphi) \end{Vmatrix} \tag{6.1}$$

They can then be written in the form

$$\nabla Y^* = \frac{1}{\sin\theta}\begin{Vmatrix} (\sin^2\theta Z)^{\cdot} \\ Z^{\cdot} \end{Vmatrix} \tag{6.2}$$

If the conditions for the displacements on the surfaces $\varphi = \varphi_i$, $\theta = \omega_i$ ($i = 0, 1$) to be zero are satisfied, we must apply the following boundary conditions to Eqs (6.2) (everywhere henceforth $i = 0, 1$; $n = 1, 2, \ldots$; $k = 0, 1, 2, \ldots$)

$$Y^*(r,\theta,\varphi_i)=0, \quad \varphi_0=0; \quad \omega_0\leqslant\theta\leqslant\omega_1; \quad Y^*(r,\omega_i,\varphi)=0, \quad 0\leqslant\varphi\leqslant\varphi_1 \tag{6.3}$$

We first apply integral transformation (3.5) to boundary-value problems (6.2) and (6.3), i.e.

$$Y_n^*(r,\theta)=\int_0^{\varphi_1} Y^*(r,\theta,\varphi)\sin\mu_n\varphi d\varphi, \quad \mu_n=\frac{n\pi}{\varphi_1} \tag{6.4}$$

We thereby obtain

$$-\nabla_n^* Y_n^*=\frac{1}{\sin\theta}\begin{Vmatrix}(\sin^2\theta Z_n)^{\cdot}\\ -\mu_n Z_n^c\end{Vmatrix}, \quad Z_n^c=\int_0^{\varphi_1} Z\cos\mu_n\varphi d\varphi; \quad Y_n^*(r,\omega_i)=0 \tag{6.5}$$

(the functions Z_n are given by (3.5)).

We will use integral transformation (4.10) to solve boundary-value problems (6.5). We thereby obtain

$$-Y_{nk}^*(r)=\frac{1}{N_\nu}\int_{\omega_0}^{\omega_1}\begin{Vmatrix}\sin^2\theta Z_n y_0^{\cdot}(\theta,\nu)\\ -\mu_n Z_n^c y_0(\theta,\nu)\end{Vmatrix}d\theta, \quad \nu=\nu_k^c \tag{6.6}$$

The formula for the transform $u_{nk}(r)$ follows from relations (5.24) and (5.8), in which we must put $\nu = \nu_k^0$, $\mu = \mu_n = n\pi\varphi_1^{-1}$. From the transforms u_{nk}, V_{nk}^* and W_{nk}^*, using inversion formulae (4.13) and then (3.6), we obtain $U(r, \theta, \varphi)$, $V^*(r, \theta, \varphi)$, $W^*(r, \theta, \varphi)$. As a result, taking relations (6.1) and (2.1) into account we obtain the displacement field for the problem in explicit form, when the displacements are specified on the surfaces $\varphi = \varphi_i$ and $\theta = \omega_i$.

For the case when $\varphi_0 = -\pi$ and $\varphi_1 = \pi$ in conditions (1.1), while, as previously, the displacements are specified on the surface $\theta = \omega_i$, the formulae obtained must be corrected as follows. The transform $u_{nk}(r)$, as previously, is found from formulae (5.24) and (5.8), in which $\nu = \nu_k^a$ and $\mu = |n|$. When calculating V_{nk}^* and W_{nk}^*, instead of transform (6.4) we must take

$$Y_n^*(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} Y^*(r,\theta,\varphi)e^{-in\varphi}d\varphi \tag{6.7}$$

Then, in Eq. (6.5) we must take $-in$ instead of μ_n, while the transforms Z_n and Z_n^c are replaced by transform Z_n, defined by (3.13). We must also make this change in formulae (6.6), additionally replace $y_0(\theta, \nu)$ by $\varphi_a^m(\theta, \nu)$ and take into account that $\nu = \nu_k^a$. From the transforms obtained u_{nk}, V_{nk}^*, W_{nk}^*, using inversion formulae (4.18) and then (3.14), we obtain their originals and thereby obtain explicit formulae for the required displacement field.

We will now consider the case of sliding clamping, when conditions (1.2) and (1.3) must be satisfied. If we take notation (2.1) and (6.1) into account, conditions (1.3) can be written in the form

$$\begin{aligned}&W^*(r,\theta,\varphi_i)=0, \quad \omega_0\leqslant\theta\leqslant\omega_1,\\ &U^{\cdot}(r,\theta,\varphi_i)+r(W^*(r,\theta,\varphi_i)r^{-1})'=0\\ &\sin\theta[\sin^{-2}\theta W^*(r,\theta,\varphi_i)]^{\cdot}+\sin^{-2}\theta V^{*\cdot}(r,\theta,\varphi_i)=0, \quad \omega_0\leqslant\theta\leqslant\omega_1\end{aligned} \tag{6.8}$$

Since the condition $U^{\cdot}(r, \theta, \varphi_i) = 0$ is already satisfied by virtue of integral transformation (3.9), conditions (1.3) will be completely satisfied if, in addition to conditions (6.8), the following condition is satisfied

$$V^{*\cdot}(r,\theta,\varphi_i)=0, \quad \omega_0\leqslant\theta\leqslant\omega_1 \tag{6.9}$$

In the same notation, conditions (1.2) can be written in the form

$$V^*(r,\omega_i,\varphi)=0, \quad \varphi_0\leqslant\varphi\leqslant\varphi_1 \tag{6.10}$$

$$U'(r,\omega_i,\varphi)+r^2[r^{-1}V^*(r,\omega_i,\varphi)]'\operatorname{cosec}\omega_i=0$$

$$W^{*\prime}(r,\omega_i,\varphi)-2W^*(r,\omega_i,\varphi)\operatorname{ctg}\omega_i+V^{*\prime}(r,\omega_i,\varphi)\operatorname{cosec}\omega_i=0 \tag{6.11}$$

Since the condition $U'(r,\ \omega_i,\ \varphi)\ =\ 0$ is already satisfied by virtue of integral transformation (4.17), conditions (1.2) will be completely satisfied if, in addition to condition (6.10) the following equality is satisfied

$$W^{*\prime}(r,\omega_i,\varphi)-2W^*(r,\omega_i,\varphi)\operatorname{ctg}\omega_i=0,\qquad \varphi_0\leqslant\varphi\leqslant\varphi_1 \tag{6.12}$$

Hence, we must add boundary conditions (6.8)–(6.10) and (6.12) to differential equations (6.2).

To solve the above boundary-value problems ($\varphi_0 = 0$) we will use integral transformations (3.9) and (3.5). As a result, these boundary-value problems are converted into the following one-dimensional problems

$$-\nabla_n^* V_n^*=\operatorname{cosec}\theta(\sin^2\theta Z_n)',\quad \mu_n=(n-1)\pi\varphi_i^{-1},\quad V_n^*(r,\omega_i)=0 \tag{6.13}$$

$$\nabla_n^* W_n^*=\mu_n\int_0^{\varphi_1} Z\cos\mu_n\varphi d\varphi,\quad \mu_n=\frac{n\pi}{\varphi_1},\quad W_n^{*\prime}(r,\omega_i)-2W_n^*(r,\omega_i)\operatorname{ctg}\omega_i=0 \tag{6.14}$$

(the expression for the function Z_n is taken according to integral transformation (3.9)).

We will solve boundary-value problem (6.13) using integral transformation (4.6) with $j = 0$. As a result we obtain

$$V_{nk}^*(r)=-\frac{1}{N_\nu}\int_{\omega_0}^{\omega_i} Z_n(r,\theta)y_0'(\theta,\nu)\sin^2\theta d\theta,\qquad \nu=\nu_k^0 \tag{6.15}$$

To solve boundary-value problem (6.14) we will use integral transformation (4.6) with $j = 1$, where we put $h_i = -2\operatorname{ctg}\omega_i$ and $\mu = \mu_n = n\pi\varphi_1^{-1}$ in formulae (4.3), (4.4) and (4.9), which define the eigenfunctions and eigenvalues ν_k^1. This must also be done in the inversion formula

$$W_n^*(r,\theta)=-\sum_{k=0}^{\infty}\frac{W_{nk}^*(r)y_1(\theta,\nu_k^1)}{\sigma_{\mu k}^1(\omega_0,\omega_1)} \tag{6.16}$$

Application of integral transformation (4.6) with $j = 1$ to boundary-value problem (6.14) leads to the formula

$$W_{nk}^*(r)=-\frac{\mu_n}{N_\nu}\int_{\omega_0}^{\omega_1}\int_0^{\varphi_1} Z(r,\theta,\varphi)\cos\mu_n\varphi y_1(\theta,\nu_k^1)\sin\theta d\theta d\varphi$$

We obtain the transform $u_{nk}(r)$, as previously, from formulae (5.24) and (5.8), in which we must take the following values for the parameters μ and ν

$$\mu=\mu_n=(n-1)\pi\varphi_1^{-1},\qquad \nu=\nu_k^*$$

If the transforms $u_{nk}(r)$, $V_{nk}(r)$ and $W_{nk}(r)$ obtained are inverted using the inversion formulae (4.18), (4.13) and (6.16), and also (3.10) and (3.6), we obtain the exact solution of the problem in the case of sliding clamping along the surfaces $\varphi = \varphi_i$ and $\theta = \omega_i$.

When, under conditions (1.1), $\varphi_0 = -\pi$, $\varphi_1 = \pi$ and on the surfaces $\theta = \omega_i$ sliding clamping occurs, we must introduce these corrections into the solution obtained. Instead of integral transformations (3.9) and (3.5) we must take integral transformation (6.7). As a result, Eqs (6.2) become Eqs (6.5), in which we must take $-in$ instead of μ_n, while Z_n and Z_n^c must replace the transform Z_n, defined by (3.13). The boundary conditions for the differential equations obtained, by relations (6.10) and (6.11), are written in the form

$$V_n^*(r,\omega_i)=0,\qquad W_n^{*\prime}(r,\omega_i)-2W_n^*(r,\omega_i)\operatorname{ctg}\omega_i=0$$

These boundary conditions necessitate that the following integral transformations must be applied to the boundary-value problems obtained instead of integral transformation (4.6) when $j = 0$ and $j = 1$

$$Y_{nk}^*(r) = \int\limits_{\omega_0}^{\omega_1} \left\| \begin{matrix} V_n^*(r,\theta)\varphi_a^m(\theta,\nu_k^a) \\ W_n^*(r,\theta)\varphi_b^m(\theta,\nu_k^b) \end{matrix} \right\| \sin\theta d\theta, \qquad m = |n| \tag{6.17}$$

The expressions for the eigenfunctions $\varphi_a^m(\theta, \nu_k^a)$ and $\varphi_b^m(\theta, \nu_k^b)$ and the equations for determining the eigenvalues ν_k^a, ν_k^b are given by (2.24), (2.27) and (2.31), (2.34) from [5], where, in the last two, we must put

$$h_i = -2\operatorname{ctg}\omega_i \tag{6.18}$$

The inversion formulae for transforms (6.17) have the form [5]

$$Y_n(r,\theta) = -\sum_{k=0}^{\infty} \frac{\varphi_e^m(\theta,\nu_k^e)}{\sigma_{mk}^e(\omega_0,\omega_1)} \left\| \begin{matrix} V_{nk}(r), & e = a \\ W_{nk}(r), & e = b \end{matrix} \right\| \tag{6.19}$$

The expression for $\sigma_{mk}^a(\omega_0, \omega_1)$ is given by (2.28) from [5], while $\sigma_{mk}^b(\omega_0, \omega_1)$ is given by (2.40) from [5], in which we must take relation (6.18) into account.

The application of integral transformation (6.17) to the above-mentioned boundary-value problems for $V_n^*(r, \theta)$ and $W_n^*(r, \theta)$ enabled us to obtain their transforms in the form

$$Y_{nk}^*(r) = \frac{1}{N_\nu} \int\limits_{\omega_0}^{\omega_1} Z_n(r,\theta) \left\| \begin{matrix} \sin^2\theta\varphi_a^m(\theta,\nu), & \nu = \nu_k^a \\ \sin^2\theta\varphi_b^m(\theta,\nu), & \nu = \nu_k^b \end{matrix} \right\| d\theta$$

We obtain the transform $U_{nk}(r)$, as previously, from formulae (5.24) and (5.8), in which we must take $\mu = |n|$ and $\nu = \nu_k^c$. From the transforms obtained $U_{nk}(r)$, $V_{nk}^*(r)$, $W_{nk}^*(r)$, using inversion formulae (4.18), (6.19) and (3.14), we obtain their originals and thereby obtain explicit formulae for the displacement field.

Hence, we have obtained exact solutions for all versions of the problems. As previously [5], we can mention numerous special cases of the problems solved here.

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